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COMMENT

Level repulsion in the spectrum of two-dimensional harmonic oscillators

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Abstract. The asymptotic-energy limit of the density $P(S)$ of spacings between adjacent levels of the two-dimensional harmonic oscillator (TDHO) spectrum is studied. It is shown that in any integer segment $[M, M+1]$, containing $\sim M/\alpha$ levels, of the TDHO spectrum $m+\alpha n$, $P(S)$ has the form $\Sigma\ w_i\delta(S-S_i)$ where i takes on at most three values. For large M, $P(S)$ displays strong level repulsion for irrational α, but it does not settle on a stationary form nor does its average over M. This is in marked contrast with the behaviour of generic integrable systems for which the Poisson statistics, $P(S)=\exp(-S)$, is known to apply.

In recent years the energy level fluctuations (i.e. departures from uniformity) in the spectra of bound systems with more than one degree of freedom have been studied extensively. The underlying belief [1] is that the fluctuations at asymptotic energies should depend on whether the corresponding classical motion is integrable or chaotic. The proposition [1, 2] is that in the integrable case the levels are distributed randomly as a Poisson process, whereas in the chaotic case they follow, as in complex nuclear spectra, the eigenvalue distribution of certain random matrices. However, the simplest integrable systems, namely harmonic oscillators, provide the major exception [1]. The purpose of this comment is to give exact analytic results for the system of two-dimensional harmonic oscillators (TDHO).

For chaotic systems, recent studies [2] show that the spectrum is a rather rigid one, having small fluctuations. The rigidity has short-range features (repulsion among neighbouring levels) as well as long-range (significant correlation among distant levels). The standard random-matrix model for such systems with time-reversal invariance is the Gaussian orthogonal ensemble of real symmetric matrices and, without invariance, the Gaussian unitary ensemble of complex Hermitian matrices. For integrable systems Berry and Tabor [1] have shown that the Poisson results apply in the generic case where the energy contours in action space are curved. They also observed that the harmonic oscillators are non-generic (with flat energy contours) and have anomalous fluctuations. Their numerical experiments demonstrated that, when the frequencies are incommensurable, the spectrum displays strong level repulsion, similar qualitatively to the chaotic case and contrasting sharply with the level clustering of the generic

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integrable case. The present study of the TDHO spectrum pursues their suggestion that the source of level repulsion may lie in the nature of the best rational approximations of the frequency ratio which are given by its simple continued fraction [3] (SCF). The derivations are rather involved and will be given elsewhere; but the final results provide a pleasing picture which we present here.

Without loss of generality, we write the TDHO levels as $E_{mn} = m + \alpha n$ where m, $n = 0, 1, 2, \ldots$, and α (>0) is the frequency ratio. (A rational α allows degenerate levels because $m + \alpha n = (m \pm p) + \alpha(n \mp q)$ for $\alpha = p/q$ with p, q relative primes, but an irrational α does not.) Consider the unit-length segment of this spectrum contained in $[M, M+1]$, with M an integer, including the end points; in case of degeneracies only one level ($m = M+1$, $n = 0$) at the upper end is included; thus adjacent segments contain one level in common, ensuring a complete counting of the nearest-neighbour spacings. Let us use the notation that $\mathrm{Int}(X)$ is the largest integer $\leq X$ and $\mathrm{Frac}(X) = X - \mathrm{Int}(X) = X \pmod 1$. Then the segment contains a total of $(N+1)$ levels with average spacing N^{-1} where $N = -\mathrm{Int}(-(M+1)/\alpha) \simeq M/\alpha$; in other words the average level density increases linearly with energy ($\rho_{\mathrm{av}}(E) \simeq E/\alpha$ for large E). It is also easy to show that the levels are located at $M + x^{(n)}$ where

$$x^{(n)} = \begin{cases} \mathrm{Frac}(n\alpha) & n = 0, 1, \ldots, N-1 \\ 1 & n = N. \end{cases} \tag{1}$$

Ignoring the constant M, we refer to the $x^{(n)}$ as the levels of the segment and denote the corresponding ordered spectrum by $\{x_l\}$ where $0 = x_0 \leq x_1 \leq \ldots \leq x_{N-1} < x_N = 1$. The spectrum is constructed below explicitly by finding the (integer) ordering function X_l, defined, for $0 \leq l < N$ and $0 \leq X_l < N$, by $x_l = \mathrm{Frac}(X_l \alpha)$.

We find that, for given N, the ordering function is the same for a range of α values and the range is such that, when α varies, the level motion is strongly correlated. A remarkable aspect of this level correlation is that there are *at most three distinct nearest-neighbour spacings* for any α, rational or irrational, and for any $N \geq 1$.

This result follows easily for rational $\alpha = p/q$ when $N \geq q$. For, in this case the $x^{(n)}$ are integer multiples of q^{-1}, and $x^{(n)} = x^{(n')}$ if and only if $n - n' = 0 \pmod q$. Thus the spectrum is uniformly spaced (one distinct spacing) when $N = q$, and the same with degeneracies (two distinct spacings) when $N > q$. The spacing density $P_N(S)$ of the segment, normalised to unit integral and unit average spacing, is given as

$$P_N\left(S; \alpha = \frac{p}{q}, N \geq q\right) = \left(1 - \frac{q}{N}\right)\delta(S) + \frac{q}{N}\,\delta\left(S - \frac{N}{q}\right) \xrightarrow{N \to \infty} \delta(S) \tag{2}$$

where the last form does not give the correct average spacing but emphasises the dominance of degeneracies at asymptotic energies. For other n values and other α values the three-spacing result can be guessed easily from numerical experiments, for example by realising the segment on a circle of unit length where $x^{(0)}$ and $x^{(N)}$ coincide ($x^{(n)} \equiv (2\pi)^{-1} \exp(2\pi n\alpha \mathrm{i})$ for $n = 0, 1, \ldots, N-1$; see figure 1).

For a rigorous derivation, consider the SCF of α:

$$\begin{aligned} \alpha &= a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ldots}} \\ &= [a_0; a_1, \ldots, a_k, r_{k+1}] \\ &= [a_0; a_1, a_2, \ldots] \end{aligned} \tag{3}$$

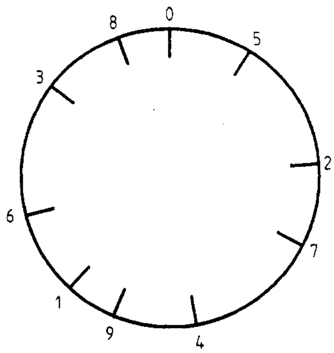

Figure 1. The $x^{(n)}$ sequence, realised on a circle, for $n = 0, 1, \ldots, 9$ and $\alpha = (\sqrt{5}-1)/2$.

where $a_k = \text{Int}(r_k)$ is the kth ($k \geq 0$) element of the SCF and $r_k = [a_k; a_{k+1}, a_{k+2}, \ldots]$ the kth remainder, such that $a_k > 0$ for $k > 0$. Its kth convergent is a rational fraction, given by $p_k/q_k = [a_0; a_1, \ldots, a_k]$ where p_k and q_k, relative primes, can be obtained recursively from $p_k = a_k p_{k-1} + p_{k-2}$ and $q_k = a_k q_{k-1} + q_{k-2}$ with $p_{-1} = 1$, $q_{-1} = 0$, $p_0 = a_0$ and $q_0 = 1$. The convergents provide an alternating sequence of lower and upper bounds on α, and are the best rational approximations to α in the sense that $|\alpha d - c|$ is a minimum whenever the rational c/d is equal to a convergent [3]; the minimum values are given by

$$t_k = (-1)^k(\alpha q_k - p_k) = \frac{1}{q_{k-1} + q_k r_{k+1}} \leq \frac{1}{q_{k+1}}. \tag{4}$$

The intermediate fractions [3] $(p_{k-1} + rp_k)/(q_{k-1} + rq_k) = [a_0; a_1, \ldots, a_k, r]$, for $r = 1, 2, \ldots, a_{k+1} - 1$, which form a montonic sequence between p_{k-1}/q_{k-1} (i.e. $r = 0$) and p_{k+1}/q_{k+1} (i.e. $r = a_{k+1}$), also provide approximations to α (though not the best ones in the sense defined above); in this case the errors are determined by

$$\tilde{t}_k(r) = (-1)^{k+1}(\alpha(q_{k-1} + rq_k) - (p_{k-1} + rp_k)) = (r_{k+1} - r)t_k \tag{5}$$

valid for $r = 0$, a_{k+1} also.

Now let us write $N = q_{k-1} + rq_k + s$ where $k \geq 0$, $1 \leq r \leq a_{k+1}$ and $0 \leq s < q_k$, obtaining a unique representation of N in terms of the integers k, r and s. We find then that X_l is given, for $0 \leq l < N$, by

$$X_l = (-1)^k(\lambda_l q_k - \mu_l(q_{k-1} + rq_k) - \nu_l(q_{k-1} + (r-1)q_k)). \tag{6}$$

Here λ_l, μ_l and ν_l are non-negative integer functions of N whose sum is l, and which are non-decreasing with l: $0 = \lambda_0 \leq \lambda_1 \leq \ldots \leq \lambda_{N-1} \leq \lambda_N = (N - q_k)$, $0 = \mu_0 \leq \mu_1 \leq \ldots \leq \mu_{N-1} \leq \mu_N = s$, $0 = \nu_0 \leq \nu_1 \leq \ldots \leq \nu_{N-1} \leq \nu_N = (q_k - s)$; i.e. as l increases by unity, only one of the increments, $(\lambda_{l+1} - \lambda_l)$, $(\mu_{l+1} - \mu_l)$ and $(\nu_{l+1} - \nu_l)$, becomes unity while the other two remain zero. The definition is completed by requiring that the unit increments in λ_l and μ_l take priority over that in ν_l, such that, after the increment, the expression in (6) satisfies $0 \leq X_{l+1} < N$. Thus $(X_{l+1} - X_l)$ is either $(-1)^k q_k$ or $(-1)^{k+1}(q_{k-1} + rq_k)$ or $(-1)^{k+1}(q_{k-1} + (r-1)q_k)$, and acquires no other value. The $l = N$ case does not apply to X_l—mere substitution yields $X_N = 0$—but it does to x_l in (7) below, where $x_N = 1$. Equations (4)-(6), along with the definition of X_l, yield the ordered spectrum as

$$x_l = \lambda_l t_k + \mu_l \tilde{t}_k(r) + \nu_l \tilde{t}_k(r-1) \tag{7}$$

with $0 \leqslant l \leqslant N$. Remarkably, (6), (7) are valid for all $\alpha = (p_{k-1} + \eta p_k)/(q_{k-1} + \eta q_k) = [a_0; a_1, \ldots, a_k, \eta]$ where $\eta \geqslant r$ if we replace r_{k+1} by η in t_k and $\tilde{t}_k$ of (4), (5); thus alternative expressions for λ_l, μ_l and ν_l can be given in terms of the spectrum for special values of η, in particular $\eta = r+1$. We remark also that (7) can be used to rederive the familiar result that when α is irrational the $x^{(n)}$ fill the segment densely and uniformly.

The nearest-neighour spacing $(x_{l+1} - x_l)$ is either t_k or $\tilde{t}_k(r)$ or $\tilde{t}_k(r-1) = \tilde{t}_k(r) + t_k$, which occur λ_N, μ_N and ν_N times respectively and, of the three, the smallest spacing is t_k when $r < a_{k+1}$, and $\tilde{t}_k(r)$ when $r = a_{k+1}$. Thus the spacing density $P_N(S)$ is given by

$$NP_N(S; \alpha) = (N - q_k)\delta(S - Nt_k) + s\delta(S - N\tilde{t}_k(r)) + (q_k - s)\delta(S - N\tilde{t}_k(r-1)). \qquad (8)$$

As a trivial example, we have $P_1(S) = \delta(S-1)$ for $N = 1$. As a non-trivial example we obtain (2), because, in this case, $q_k = q$, $t_k = 0$, and $\tilde{t}_k(r) = \tilde{t}_k(r-1) = q^{-1}$. In all other cases we have three distinct spacings, but with zero weight for one of them when $s = 0$ (or q_k). In particular, for irrational α the distinct spacings as well as their weights do not approach a limit as $N \to \infty$, showing that at asymptotic energies the segment does not have a stationary spacing distribution. Moreover, the spacing density $\bar{P}(S)$ of the segment $[M_{\min}, M_{\max}+1]$, $\bar{P}(S) = \Sigma\, NP_N(S)/\Sigma\, N$ where N takes on appropriately chosen values such that all values of M between $M_{\min} \simeq \alpha N_{\min}$ and $M_{\max} \simeq \alpha N_{\max}$ are realised, also does not have a proper limit as $M_{\max} \to \infty$, changing continually from one pattern to another without settling on a stationary form as the energy $M_{\max}$ increases.

In general $\bar{P}(S)$ for large $M_{\max}$ depends on three parameters, L, p and ε, defined, as above, by $N_{\max} \simeq q_{L-1} + pq_L + \varepsilon q_L$ where p takes on integer values between 1 and a_{L+1}, and ε varies continuously between 0 and 1 for large L. As a consequence of taking ε as a continuous variable, we end up for $\bar{P}(S)$ with a relatively smooth density rather than a sum of δ functions. Simplifications occur when α is a quadratic irrational (root of a quadratic equation with integer coefficients). In this case the a_L sequence is periodic for large L, so that $\bar{P}(S)$ as a function of L is also periodic for large L (and hence recurrent as a function of $M_{\max}$), the period being the same as that of a_L. When the period is unity, i.e. $a_L \to a$ (an integer >0) for large L, the $L \to \infty$ limit exists. We find, in terms of the other two parameters, that

$$\bar{P}(S; \varepsilon, p) = (\Phi(\varepsilon, p))^{-1}[\bar{P}_{\text{II}}(S; 1, a) + (\gamma^2 - 1)\bar{P}_{\text{II}}(s; \varepsilon, p)]$$

$$\bar{P}_{\text{II}}(S; \varepsilon, p) = \sum_{r=1}^{p} \bar{P}_{\text{I}}(S; 1, r) + \bar{P}_{\text{I}}(S; \varepsilon, p) - \bar{P}_{\text{I}}(S; 1, p)$$

$$\begin{aligned}\bar{P}_{\text{I}}(S; \varepsilon, p) = {} & (\gamma + \gamma^{-1})^2\left(S - \frac{1}{\gamma + \gamma^{-1}}\right) h\left(S; \frac{p + \gamma^{-1}}{\gamma + \gamma^{-1}}, \frac{\varepsilon + p + \gamma^{-1}}{\gamma + \gamma^{-1}}\right) \\ & + \left(\frac{\gamma + \gamma^{-1}}{\gamma - p}\right)^2 \left(S - \frac{(\gamma - p)(p + \gamma^{-1})}{\gamma + \gamma^{-1}}\right) \\ & \times h\left(S; \frac{(\gamma - p)(p + \gamma^{-1})}{\gamma + \gamma^{-1}}, \frac{(\gamma - p)(\varepsilon + p + \gamma^{-1})}{\gamma + \gamma^{-1}}\right) \\ & + \left(\frac{\gamma + \gamma^{-1}}{\gamma - p + 1}\right)^2 \left(\frac{(\gamma - p + 1)(1 + p + \gamma^{-1})}{\gamma + \gamma^{-1}} - S\right) \\ & \times h\left(S; \frac{(\gamma - p + 1)(p + \gamma^{-1})}{\gamma + \gamma^{-1}}, \frac{(\gamma - p + 1)(\varepsilon + p + \gamma^{-1})}{\gamma + \gamma^{-1}}\right)\end{aligned}$$

$$h(S;\, S_1, S_2) = \begin{cases} 1 & S_1 < S < S_2 \\ \frac{1}{2} & S = S_1 \text{ or } S_2 \\ 0 & S < S_1 \text{ or } S > S_2 \end{cases}$$

$$\Phi(\varepsilon, p) = (\gamma^{-1} + \tfrac{1}{2})a + \tfrac{1}{2}a(a+1) + (\gamma^2 - 1)$$
$$\times [(\gamma^{-1} + \tfrac{1}{2})(p-1) + \tfrac{1}{2}p(p-1) + (\gamma^{-1} + p)\varepsilon + \tfrac{1}{2}\varepsilon^2]$$

$$\gamma \equiv \gamma(a) = \tfrac{1}{2}[a + (a^2+4)^{1/2}] = [a;\, a, a, \ldots]. \qquad (9)$$

Note that $\bar{P}(S;\, 1, p) = \bar{P}(S;\, 0, p(\text{mod } a) + 1)$. See figure 2 for illustrations of (9) with $a = 1, 2$.

It is easily seen from (9) that the non-zero values of $\bar{P}(S)$ occur for $S > (a^2+4)^{-1/2}$. This implies strong level repulsion in the spectrum, the strongest being obtained for $a = 1$. Other irrational α values also exhibit level repulsion, which is strongest when poorest rational approximations to α occur. Other fluctuation measures have not been analysed in detail. But the essential conclusions for the irrational case are the same: no stationary statistics at asymptotic energies and, in general, strong spectral rigidity of the short-range as well as long-range kind.

In conclusion, we stress again that the TDHO spectrum is very different from the spectrum of a generic integrable system as well as of a chaotic system because of the existence of at most three distinct spacings in unit-length segments. Fluctuation properties at asymptotic energies do not settle on a stationary behaviour. However, the irrational TDHO does exhibit level repulsion which in general is much stronger than

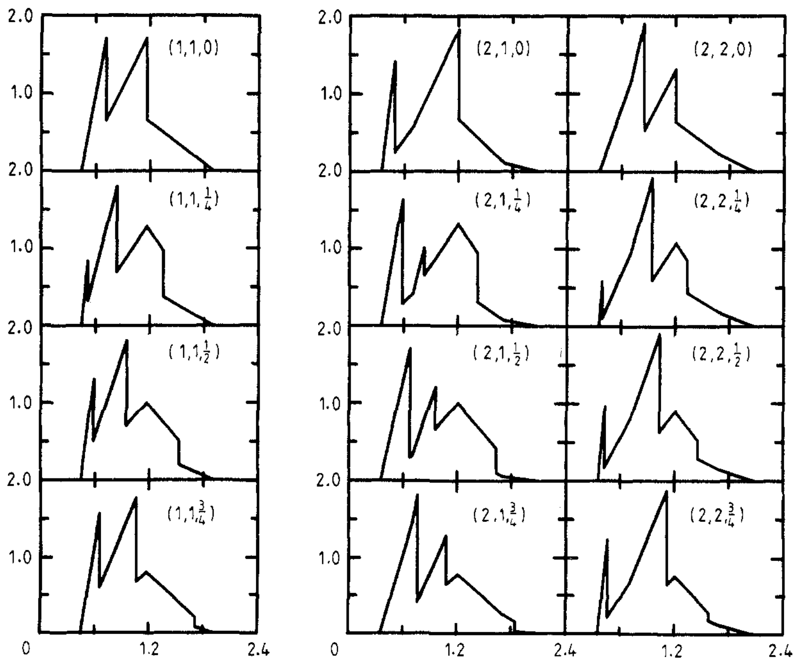

Figure 2. $\bar{P}(S;\, \varepsilon, p)$ of (9) plotted as against S for $a = 1, 2$, applying, for example, for $\alpha = (\sqrt{5}-1)/2 = [0;\, 1, 1, \ldots]$ and $\alpha = 1/\sqrt{2} = [0;\, 1, 2, \ldots]$ respectively. Each box gives (a, p, ε) in its upper right corner. As an application, consider the first 5000 levels of $\alpha = 1/\sqrt{2}$, corresponding to $L = 5$, $p = 2$ and $\varepsilon = 0.5$; even though L is small, the histogram of $\bar{P}(S)$ for (2, 2, 0.5) is found to agree extremely well with figure 5(a) of [1].

that in the chaotic case. The anomalous fluctuations in higher dimensions are not known, but one would expect non-stationarity and level repulsion to extend.

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